Solution 11

1. Determine which of the following sets are dense, open dense, nowhere dense, of first category and residual in $\mathbb{R}$ (you may draw a table):

   (a) $A = \{n/2^m : \ n, m \in \mathbb{Z}\}$,
   (b) B, all irrational numbers,
   (c) $C = \{0, 1, 1/2, 1/3, \cdots\}$,
   (d) $D = \{1, 1/2, 1/3, \cdots\}$,
   (e) $E = \{x : \ x^2 + 3x - 6 = 0 \}$,
   (f) $F = \cup_k (k, k+1), k \in \mathbb{N}$,

   Solution. (a) A is dense, not open, not nowhere dense, of first category and not residual.

   (b) B is dense, not open, not nowhere dense, of second category and residual.

   (c) C is not dense, not open (closed in fact), nowhere dense, of first category and not residual.

   (d) D is not dense, not open (not closed), nowhere dense, of first category and not residual.

   (e) E is the finite set $\{(-3+\sqrt{33})/2, (-3-\sqrt{33})/2\}$. It is not dense, not open (closed in fact), nowhere dense, of first category and not residual.

   (e) F is dense, open, not nowhere dense, of second category and residual.

Sets	Dense	Open dense	Nowhere dense	First category	Residual
A	✓	✗	✗	✓	✗
B	✓	✗	✗	✗	✓
C	✗	✗	✓	✓	✗
D	✗	✗	✓	✓	✗
E	✗	✗	✓	✓	✗
F	✓	✓	✗	✗	✓

2. Determine which of the following sets are dense, open dense, nowhere dense, of first category and residual in $C[0,1]$ (you may draw a table):

   (a) $\mathcal{A}$, all polynomials whose coefficients are rational numbers,
   (b) $\mathcal{B}$, all polynomials,
   (c) $\mathcal{C} = \{f : \ \int_0^1 f(x)dx \neq 0\}$,
   (d) $\mathcal{D} = \{f : \ f(1/2) = 1 \}$.

   Solution. (a) $\mathcal{A}$ is dense (and countable too), not open, not nowhere dense, of first category, and not residual.

   (b) $\mathcal{B}$ is dense (and uncountable), not open, not nowhere dense, of first category and not residual. ($\mathcal{B}$ can be expressed as the countable union of P_n where P_n is the set of all polynomials of degree not exceeding n. Each P_n is closed and nonwhere dense.)

   (c) $\mathcal{C}$ is dense, open, not nowhere dense, of second category, and residual.

   (d) $\mathcal{D}$ is not dense, not open (closed in fact), nowhere dense, of first category, and not residual.

Sets	Dense	Open dense	Nowhere dense	First category	Residual
$\mathcal{A}$	✓	✗	✗	✓	✗
$\mathcal{B}$	✓	✗	✗	✓	✗
$\mathcal{C}$	✓	✓	✗	✗	✓
$\mathcal{D}$	✗	✗	✓	✓	✗

3. Use Baire category theorem to show that transcendental numbers are dense in the set of real numbers.

Solution. A number is called algebraic if it is a root of some polynomial with integer coefficients and it is transcendental otherwise. Let $\mathcal{A}$ be all algebraic numbers and $\mathcal{T}$ be all transcendental numbers so that $\mathbb{R} = \mathcal{A} \cup \mathcal{T}$. We know that $\mathcal{A}$ is a countable set $\{a_j\}$. Thus let $\mathcal{A}_n = \{a_1, \cdots, a_n\}$ and we have $\mathcal{T} = \cap_n \mathbb{R} \setminus \mathcal{A}_n$. As each $\mathbb{R} \setminus \mathcal{A}_n$ is a dense, open set, $\mathcal{T}$ is a residual set and therefore dense by Baire Category Theorem.

4. A set E in a metric space is called a perfect set if, for each point $x \in E$ and $r > 0$, the ball $B_r(x) \bigcap E$ contains a point different from x.

   (a) For each x in the perfect set E, there exists a sequence in E consisting of infinitely many distinct points converging to x.

   (b) Every complete perfect set is uncountable. Hint: Use Baire Category Theorem.

   (c) Is (b) true without completeness?

Solution. (a). For each $n \geq 1$, as $(B_{1/n}(x) \setminus \{x\}) \bigcap E$ is nonempty, we pick a point from it to form $\{x_n\}$. Obviously, there are infinitely many distinct points in this sequence and it converges to x as $n \to \infty$.

(b). Assume on the contrary that the perfect set E is countable, $E = \{a_n\}, n \geq 1$. We have $E = \bigcup_{n=1}^{\infty} \{a_n\}$. Obviously every $\{a_n\}$ is a closed set. On the other hand, every ball containing a_n must contain some points different from a_n. We conclude that every $\{a_n\}$ is a closed set with empty interior. However, by assumption, (E, d) is a complete metric space. By Baire Category Theorem E cannot have such decomposition. Therefore, it must be uncountable.

Note. Applying to $\mathbb{R}$, it gives another proof that $\mathbb{R}$ is uncountable.

(c). No. Simply consider $\mathbb{Q}$ under the Euclidean metric. It is a countable perfect set which is not complete. Think of the Cauchy sequence $\{3, 3.1, 3.14, 3.141, 3.1415, 3.14159, \cdots\}$ which is in $\mathbb{Q}$ but converges to π.

5. Let $\|\cdot\|$ be a norm on $\mathbb{R}^n$.

   (a) Show that $\|x\| \leq C\|x\|_2$ for some C where $\|\cdot\|_2$ is the Euclidean metric.

   (b) Deduce from (a) that the function $x \mapsto \|x\|$ is continuous with respect to the Euclidean metric.

   (c) Show that the inequality $\|x\|_2 \leq C'\|x\|$ for some C' also holds. Hint: Observe that $x \mapsto \|x\|$ is positive on the unit sphere $\{x \in \mathbb{R}^n : \|x\|_2 = 1\}$ which is compact.

   (d) Establish the theorem asserting any two norms in a finite dimensional vector space are equivalent.

Solution. (a). Let $x = a_1 e_1 + \cdots + a_n e_n$. By Cauchy-Schwarz Inequality

$$\|x\| = \|\sum_k a_k e_k\| \leq \sum_k |a_k| \ \|e\|_k \leq C\|x\|_2 \ ,$$

where

$$C = \sqrt{\sum_k \|e_k\|^2}\ .$$

(b). Let $x_n \to x$ in $\|\cdot\|_2$, that is, $\|x_n - x\|_2 \to 0$. By (a), $\|x_n - x\| \to 0$ too.

(c). The map $x \mapsto \|x\|$ is continuous and positive on the unit sphere. As the sphere is compact, it has a positive lower bound, that is, $\|x\| \geq \rho > 0$ whenever $\|x\|_2 = 1$. Now, given any non-zero vector x, $x/\|x\|_2$ belong to the unit sphere, so

$$\left\| \frac{x}{\|x\|_2} \right\| \geq \left\| \frac{x}{\|x\|_2} \right\|_2 \geq \rho\ .$$

(d). Let $\|\cdot\|_a$ and $\|\cdot\|_b$ be two norms on the finite dim space V. Fix a basis $\{v_1, \cdots, v_n\}$ in V. Every vector x has a unique representation $x = \sum_{k=1}^n a_k v_k$. The map $x \mapsto (a_1, \cdots, a_n)$ is a linear bijection (linear isomorphism) from V to $\mathbb{R}^n$. It induces two norms on $\mathbb{R}^n$ by $\|a\|_a = \|\sum_k a_k v_k\|_a$ and $\|a\|_b = \|\sum_k a_k v_k\|_b$ (using the same notations). From (c) both are equivalent to the Euclidean norm, hence they are also equivalent to each other. Going back to V, we conclude that they are equivalent too.

6. Let $\mathcal{F}$ be a subset of $C(X)$ where X is a complete metric space. Suppose that for each $x \in X$, there exists a constant M depending on x such that $|f(x)| \leq M,\ \forall f \in \mathcal{F}$. Prove that there exists an open set G in X and a constant C such that $\sup_{x\in G} |f(x)| \leq C$ for all $f \in \mathcal{F}$. Suggestion: Consider the decomposition of X into the sets $X_n = \{x \in X : |f(x)| \leq n,\ \ \forall f \in \mathcal{F}\}$.

   Solution. By assumption, $X = \bigcup_n X_n$. It is clear that each X_n is closed. By the completeness of X we appeal to Baire Category Theorem to conclude that there is some n_1 such that X_{n_1} has non-empty interior, call it G. Then $|f(x)| \leq n_1,\ \forall x \in G$, for all $f \in \mathcal{F}$.

7. Optional. A function is called non-monotonic if if is not monotonic on every subinterval. Show that all non-monotonic functions form a dense set in $C[a, b]$. Hint: Consider the sets

$$\mathcal{E}_n = \{f \in C[a, b] : \exists x \text{ such that } (f(y) - f(x))(y - x) \geq 0,\ \forall y,\ |y - x| \leq 1/n\}.$$

   Solution. We will show that each $\mathcal{E}_n$ is closed and . Let $f_k \to f$ uniformly and x_k satisfy $(f_k(y) - f_k(x_k))(y - x_k) \geq 0$ for $y \in [x_k - 1/n, x_k + 1/n]$. By passing to a subsequence, one may assume $x_k \to x_0$. Then

$$\left| f_k(y) - f_k(x_k) - (f(y) - f(x_k)) \right| \leq |f_k(y) - f(y)| + |f(x_k) - f_k(x_k)| \leq 2\|f_k - f\|_\infty \to 0,$$

   which shows that

$$(f(y) - f(x_0))(y - x_0) = \lim_{k\to\infty} (f(y) - f(x_k))(y - x_k) = \lim_{k\to\infty} (f_k(y) - f_k(x_k))(y - x_k) \geq 0,$$

   hence $\mathcal{E}_n$ is closed. Next, if $\mathcal{E}_n$ has non-empty interior, we can find some $f \in \mathcal{E}_n$ such that all functions in $B_\varepsilon(f)$ are in $\mathcal{E}_n$. Pick a polynomial p in $B_{\varepsilon/2}(f)$. We claim that there exists some g, $\|p - g\|_\infty \leq \varepsilon/2$, does not belong to $\mathcal{E}_n$. But $\|f - g\|_\infty < \varepsilon$, contradiction holds. Let φ be the jig-saw function that is described in our notes such that $\varphi([a, b]) = [-1, 1]$ and slope equal to a large number $\pm K$ and consider $g = p + \varepsilon/2\varphi$. Let $x \in [a, b]$ and $y > x$ close to x, we have

$$(g(y)-g(x))(y-x) = (p(y)-p(x)+\frac{\varepsilon}{2}(\varphi(y)-\varphi(x))(y-x) \leq (L(y-x)+\frac{\varepsilon}{2}(\varphi(y)-\varphi(x)))(y-x).$$

(L is a Lipschitz constant for p.) By the definition of φ, we can always choose some y close to x from the right and K so large that $L(y-x)+\varepsilon/2(\varphi(y)-\varphi(x))<0$.

It shows that $\mathcal{E}_n$ is . Similarly, let

$$\mathcal{F}_n = \{f \in C[a,b] : \exists x \text{ such that } (f(y)-f(x))(y-x) \le 0,\ \forall y,\ |y-x| \le 1/n\}.$$

Then $\mathcal{E}_n$ is closed and for all n. Let the collection of all non-monotonic functions be $\mathcal{N}$. Since a function is non-monotonic if it is either increasing or decreasing on some subinterval, we have

$$\mathcal{N} = \bigcap_n \big(C[a,b] \setminus \mathcal{E}_n \cup \mathcal{F}_n\big).$$

By Baire's theorem, $\mathcal{N}$ is a residual set and hence dense. The proof here is similar but simpler to the proof that continuous, nowhere differentiable functions form a residual set.